De Rham Cohomology:
From Complex Analysis to Algebraic Topology

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1 Motivation

Recall the following results from complex analysis:

Theorem 1.1 (Cauchy's integral theorem). *Suppose $U \subseteq \mathbb{C}$ is a simply connected open set, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let γ be a simple closed curve in U. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Theorem 1.2. *Let $U \subseteq \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$. f has a primitive on U iff for all curves γ in U, $\displaystyle\int_\gamma f(z)\,dz = 0$.*

Combining these two results, we get the following result:

Corollary 1. *Suppose $U \subseteq \mathbb{C}$ is a simply connected open set. Then any holomorphic function $f : U \to \mathbb{C}$ has a primitive.*[1]

Requiring U to be simply connected - that is, to have no holes - seems like a strange condition for holomorphic functions to have primitives. Why does it matter?

Example 1.1. Consider the holomorphic function $f(z) = \frac{1}{z}$. f is only defined on the punctured plane $\mathbb{C} \setminus \{ 0 \}$, which is not a simply-connected set. In fact, if γ is any loop winding exactly once around 0, $\int_\gamma f(z)\,dz = 2\pi i$. We could alternatively write $\operatorname{Res}(f, 0) = 1$. Regardless, by Theorem 1.2, f has no primitive on $\mathbb{C} \setminus \{ 0 \}$.

[1] Conversely, since holomorphic functions have infinitely many derivatives, functions with primitives are always holomorphic (regardless of simply-connectedness).

The punctured plane $\mathbb{C} \setminus \{ 0 \}$ has one hole in it, and $\frac{1}{z}$ is a holomorphic function on it without a primitive, meaning our assumption of simply-connectedness was necessary. But the punctured plane is *almost* simply-connected – it only has one hole! Surely there can't be too many holomorphic functions without primitives?

2 Topology of the Punctured Plane

We begin with some non-standard terminology that will simplify the discussion below.

Definition 2.1. Let $U \subseteq \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$. We say f **fails** on U if it is holomorphic but has no primitive on U.

$\frac{1}{z}$ is certainly not the only holomorphic function that fails on $\mathbb{C}\setminus\{ 0 \}$. For instance, for any constant $c \in \mathbb{C}$, $\frac{c}{z}$ fails. Furthermore, if $g : \mathbb{C}\setminus\{ 0 \} \to \mathbb{C}$ is a holomorphic function *with* a primitive, $\frac{c}{z} + g(z)$ fails. This gives us infinitely many functions that fail, but they all still somehow "use" the failure of $\frac{1}{z}$. Can't something else go wrong? Nope!

Theorem 2.1. *Suppose $f : \mathbb{C}\setminus\{ 0 \} \to \mathbb{C}$ fails. Then there exists a constant $c \in \mathbb{C}$ and a holomorphic function $g : \mathbb{C} \setminus \{ 0 \} \to \mathbb{C}$ with a primitive such that $f(z) = \frac{c}{z} + g(z)$. In fact, we have $c = \operatorname{Res}(f, 0)$.*

Proof. We need to show that $f(z) - \frac{\operatorname{Res}(f,0)}{z}$ has a primitive on $\mathbb{C} \setminus \{ 0 \}$. By Theorem 1.2, this is equivalent to showing that, for all simple closed curves γ in U, $\int_\gamma (f(z) - \frac{\operatorname{Res}(f,0)}{z})\, dz = 0$. Applying Cauchy's residue theorem, we have

$$\begin{aligned}
\int_\gamma (f(z) - \frac{\operatorname{Res}(f,0)}{z})\, dz &= \int_\gamma f(z)\, dz - \int_\gamma \frac{\operatorname{Res}(f,0)}{z}\, dz \\
&= 2\pi i \operatorname{Res}(f,0) W(\gamma,0) - 2\pi i \operatorname{Res}(f,0) W(\gamma,0) \\
&= 0,
\end{aligned}$$

where $W(\gamma, 0)$ is the winding number of γ about 0. □

In a very strong sense, this theorem says that $\frac{1}{z}$ is the "only" function that fails on $\mathbb{C}\setminus\{ 0 \}$, and the fact that there is only one such function reflects the fact that $\mathbb{C} \setminus \{ 0 \}$ has exactly one hole. To see this more precisely, we need to look at a (slightly) more complicated space.

3 Multiple Punctures in $\mathbb{C}$

Consider the multiply-punctured space $\mathbb{C} \setminus \{ p_1, \ldots, p_n \}$. Like before, all functions of the form $\frac{1}{z-p_i}$ fail, as do their nonzero linear combinations (potentially added to functions with primitives). After the last theorem, you might expect that these are the only functions that fail on this space. You would be right.

Theorem 3.1. *Suppose $f : \mathbb{C} \setminus \{ p_1, \ldots, p_n \} \to \mathbb{C}$ fails. We have $f(z) = \displaystyle\sum_{i=1}^{n} \frac{\operatorname{Res}(f, p_i)}{z - p_i} + g(z)$, where $g(z)$ is holomorphic with a primitive.*

Proof. The proof is almost identical to that of Theorem 2.1. We wish to show that $g(z) = f(z) - \displaystyle\sum_{i=1}^{n} \frac{\operatorname{Res}(f, p_i)}{z - p_i}$ has a primitive, which is iff $\int_\gamma g(z)\,dz = 0$ for every closed curve γ in U. Then, using residue theorem,

$$\begin{aligned} \int_\gamma g(z)\,dz &= \int_\gamma f(z)\,dz - \int_\gamma \sum_{i=1}^{n} \frac{\operatorname{Res}(f, p_i)}{z - p_i}\,dz \\ &= \sum_{i=1}^{n} 2\pi i \operatorname{Res}(f, p_i) W(\gamma, p_i) - \sum_{i=1}^{n} 2\pi i \operatorname{Res}(f, p_i) W(\gamma, p_i) \\ &= 0. \end{aligned}$$

□

Once again, this indicates that there are "only" n functions that fail on $\mathbb{C} \setminus \{ p_1, \ldots, p_n \}$: $\frac{1}{z-p_1}, \ldots, \frac{1}{z-p_n}$. This reflects the fact that there are n holes in our space.

4 Formalizing and Conclusion

It is useful to think about this from a slightly more formal viewpoint. Given any open set $U \subseteq \mathbb{C}$, the holomorphic functions $U \to \mathbb{C}$ form a $\mathbb{C}$-vector space, which we will call $\operatorname{Hol}(U)$. There is a subspace $P(U) \subseteq \operatorname{Hol}(U)$ consisting of all holomorphic functions on U that have primitives. What we are interested in is the *dimension of the quotient.*

Definition 4.1. The **first de Rham cohomology** of U is defined to be $H^1_{dR}(U) \coloneqq \operatorname{Hol}(U)/P(U)$.

Consider Theorems 2.1 and 3.1 in this language. What we have really shown is that $H^1_{dR}(\mathbb{C} \setminus \{ 0 \}) = \mathbb{C}$, and $H^1_{dR}(\mathbb{C} \setminus \{ p_1, \ldots, p_n \}) = \mathbb{C}^n$. The dimension of the first de Rham cohomology tells us how many holes are in the space! These holes don't have to be punctures: for example, it turns out that H^1_{dR} of $\mathbb{C}$ minus a single open ball is also $\mathbb{C}$! Cohomology is a *topological invariant*.

Why does this matter? After all, if we wanted to know how many holes were in a subset of $\mathbb{C}$, we could just...look at it. But the true power of topological machinery comes in higher dimensions. We can't look directly at an eight-dimensional sphere, for instance, but by using de Rham cohomology (in the more general language of vector fields) we could get information about its shape. Cohomology is one of the strongest tools in a topologist's toolkit, and our discussion offers only a glimpse of its power.

References and Notes

For further reading, we recommend the excellent *Differential Forms in Algebraic Topology* by Raoul Bott and Loring W. Tu. The book builds de Rham cohomology in its full glory and uses it to motivate algebraic topology more generally.

Readers familiar with basic algebraic topology may be unused to computing cohomology with differential forms. It turns out that de Rham cohomology is equivalent to singular cohomology, this is the content of **de Rham's Theorem**.

The above motivation of de Rham theory through complex analysis is, to our understanding, original. In the general theory, we are interested in integrating closed 1-forms; in our case, complex functions f where $f\,dz = 0$. Write $f = u + iv$ and note that $dz = dx + idy$. Then $f\,dz = (u\,dx - v\,dy) + (u\,dy + v\,dx)$, which equals 0 iff f is holomorphic as a complex function (using the Cauchy-Riemann equations). Thus our discussion of "holomorphic functions with primitives" is equivalent to the standard discussion of closed and exact 1-forms.